

16.8 (Stokes's Theorem)  
 16.9 (Divergence Theorem)

**Theorem (Stokes)**  
 $\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$   
 Line integration of  $\mathbf{F}$  along a curve  $C$  is equal to surface integration of  $\nabla \times \mathbf{F}$  over a surface  $S$  with boundary  $C$ .  
 Closed positively oriented curve in  $\mathbb{R}^3$ .  
 A surface with boundary  $C$ .  
 Conservative vector field  $\mathbf{F} = \nabla f$   $\Rightarrow$  integral over  $C$  is 0.

$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P dx + Q dy + R dz$   
 $= \int_a^b P(x(t), y(t), z(t)) x'(t) dt$   
 $\nabla \times (\nabla f) = 0$   
 $\nabla \times (P, Q, R) = (R_y - Q_z, P_z - R_x, Q_x - P_y)$   
 what is  $\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$  for a closed surface  $S$ ?  
 $\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \nabla \times \mathbf{F} \cdot d\mathbf{S} + \iint_{S_2} \nabla \times \mathbf{F} \cdot d\mathbf{S}$   
 $= \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$   
 $= \int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = 0$   
 Divergence Theorem:  $\iint_S \nabla \cdot (\nabla \times \mathbf{F}) dV = 0$

Example: compute  $\int_C \mathbf{F} \cdot d\mathbf{r}$  with  $\mathbf{F} = (-y^2, x, z^2)$  and  $C$  is the intersection of the plane  $y + z = 2$  and  $x^2 + y^2 = 1$

$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$   
 $\nabla \times \mathbf{F} = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix} \times \begin{pmatrix} -y^2 \\ x \\ z^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1+2y \end{pmatrix}$

On lateral boundary (cylinder), normal is perpendicular to  $\nabla \times \mathbf{F}$  b/c  $\nabla \times \mathbf{F}$  points in  $z$  direction

$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \iint_C (\nabla \times \mathbf{F}) \cdot \mathbf{n} \|\mathbf{r}_u \times \mathbf{r}_v\| dA$   
 $\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \iint_C (\nabla \times \mathbf{F}) \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \|\mathbf{r}_u \times \mathbf{r}_v\| dA = \iint_C (1+2y) \|\mathbf{r}_u \times \mathbf{r}_v\| dA$

Show this reduces to Green's theorem

$\int_C P dx + Q dy = \iint_A \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$   
 $\mathbf{r} = \langle p \cos \theta, p \sin \theta, 0 \rangle$   
 $\int_0^{2\pi} \int_0^1 (1 + p \sin \theta) p dp d\theta$

Power via Stokes

$\mathbf{r}_u = \langle -p \sin \theta, p \cos \theta, 0 \rangle$   
 $\mathbf{r}_v = \langle \cos \theta, \sin \theta, 1 \rangle$   
 $d\mathbf{S} = \mathbf{n} ds = \mathbf{n} \|\mathbf{r}_u \times \mathbf{r}_v\| dA$   
 $\mathbf{r}_u \times \mathbf{r}_v = (p \cos \theta - p \sin \theta, p \sin \theta + p \cos \theta, -p)$   
 $\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{p^2 \cos^2 \theta + p^2 \sin^2 \theta + p^2} = p \sqrt{2}$

Interpret this to give meaning of curl

$\iint_S Q_x - P_y dA$  Green's theorem!  
 $\int_C Q dx - P dy$

$\mathbf{r} = \langle r, \theta, 0 \rangle$   
 $\mathbf{e}_r = \langle \cos \theta, \sin \theta, 0 \rangle$   
 $\mathbf{e}_\theta = \langle -\sin \theta, \cos \theta, 0 \rangle$   
 $\mathbf{e}_z = \langle 0, 0, 1 \rangle$   
 $\|\mathbf{e}_r \times \mathbf{e}_\theta\| = 1$   
 $\det \begin{pmatrix} \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \|\mathbf{r}_u \times \mathbf{r}_v\| = ds$

Theorem (Divergence)

$\iiint_E \mathbf{F} \cdot d\mathbf{S} = \iiint_E \nabla \cdot \mathbf{F} dV$

Example: compute the flux of  $F = \langle z, y, x \rangle$  over the unit sphere.

**Exercises:**

- 1) Compute  $\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$  for  $F = \langle 2y \cos z, e^x \sin z, xe^y \rangle$ , on the upper half of the sphere centered at the origin with radius 9
- 2) Compute  $\int_C \mathbf{F} \cdot d\mathbf{r}$  for  $F = \langle yz, 2xz, e^{xy} \rangle$   $C$  the circle  $x^2 + y^2 = 16, z = 5$
- 3) Compute  $\iint_S \mathbf{F} \cdot d\mathbf{S}$  for  $F = \langle e^x y^2, xe^z, z^3 \rangle$  over the surface bounded by  $y^2 + z^2 = 1$  and the  $x = -1, x = 2$

4)

